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APPLICATIONS OF ALGEBRAIC GEOMETRY IN SYSTEM THEORY.(U)
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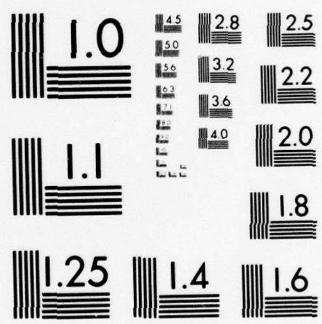

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFOSR-TR- 78-1516	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): APPLICATIONS OF ALGEBRAIC GEOMETRY IN SYSTEM THEORY	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): P.L. Falb and C.I. Byrnes / Peter L. Falb, Christopher I. Byrnes	8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3182
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Brown University, Division of Applied Math. Lefschetz Center for Dynamical Systems Providence, R.I. 02912	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 611C2F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332	12. REPORT DATE: October 1978
	13. NUMBER OF PAGES: 41 (43p.)
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

LEVEL

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

System theory is concerned with the modelling and analysis of phenomena both natural and man-made. It is a discipline whose formal beginnings go back at least to Watt and Maxwell and much of its motivation stems from engineering problems. Before World War II, a system design and analysis were primarily an art. During and after the war, techniques based on complex variable theory were developed and applied primarily to single input, single output systems represented by a rational function, called the transfer

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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20. Abstract continued.

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AFOSR-TR- 78-1516

APPLICATIONS OF ALGEBRAIC GEOMETRY

IN

SYSTEM THEORY

Dedicated to Professor Oscar Zariski on the occasion of his eightieth birthday

by

Christopher I. Byrnes* and Peter L. Falb+

October, 1978

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+Division of Applied Mathematics, Brown University, Providence, Rhode Island, 02912. This work was supported by the Air Force Office of Scientific Research under AF-AFOSR 77-3182.

1. Introduction

System theory is concerned with the modelling and analysis of phenomena both natural and man-made. It is a discipline whose formal beginnings go back at least to Watt and Maxwell and much of its motivation stems from engineering problems. Before World War II, a system design and analysis were primarily an art. During and after the war, techniques based on complex variable theory were developed and applied primarily to single input, single output systems represented by a rational function, called the transfer function. The theory of servomechanisms developed rapidly from the end of the war to the early fifties and time-domain methods were applied. The representation of transfer functions via linear, constant-coefficient, differential equations led to a renewed interest in so-called state space methods. The rapid development of the theory followed and continues today. However, the increasing complexity of the engineering and economic problems considered required greater mathematical sophistication. Little did one of the authors dream while he was a student of Professor Zariski, that the techniques of modern algebraic geometry could (and would) be applied in a critical and essential way in system theory. Our purpose, in this paper, is to illustrate several of the many applications of algebraic geometry to linear system theory. These applications are both mathematically and practically non-trivial.

We begin, in Section 2, with an analysis of two (classical) questions; namely, (i) when is a meromorphic function of the form $f(s) = \sum_{\ell=1}^{\infty} h_\ell s^{-\ell}$ rational?, and, (ii) when is such a rational function stable (i.e. has poles in the left-half plane)? The first question was answered by Hankel ([20]) and the second was answered by Cauchy, Hermite, Routh and Hurwitz ([15], [29], [30]). Our treatment serves to motivate the subsequent development. The concept of a time-invariant, finite-dimensional linear system is introduced in Section 3. Such systems admit an external description as either a matrix, $T(s)$, of rational functions or as an input-output map $f(s) = \sum_{\ell=1}^{\infty} L_\ell s^{-\ell}$ with the L_ℓ constant matrices and rank $H_f < \infty$ where $H_f = (L_{i+j-1})$ is the Hankel matrix of f. Corresponding to each external description is an internal description or representation. Thus, a pair of polynomial matrices $(R(s),P(s))$ realizes $T(s)$ if $T(s) = R(s)P^{-1}(s)$ and $R(s),P(s)$ are relatively right prime. The group $\mathcal{U}$ of polynomial matrices with determinant a unit acts on such realizations. For an input-output map f, a triple (A,B,C) of constant matrices represents f if $H_f = (CA^{i+j-1}B)$. The general linear group G (of appropriate dimension) acts on such triples via $g\cdot(A,B,C) = (gAg^{-1},gB,Cg^{-1})$. Three main topics are treated in Section 4. The first is the construction of a moduli space for linear systems under the action of G and a proof of a theorem on "realization with parameters", based on Zariski's Main Theorem and Mumford's geometric invariant theory. It turns out that the "exceptional points" are precisely the representations that are undersirable from a practical viewpoint. Thus, the mathematical and physical considerations are exactly congruent. The second topic is a brief study of the biregular and topological invariants of the moduli space with an emphasis on the impact these invariants have on system theoretic questions analogous to the questions raised in Section 2. The third topic is the consideration of moduli spaces for systems with richer symmetries such as Hamiltonian or symmetric systems. In fact, we sketch the equivalence between an old problem in network theory ([53]) and what might be called the "serre conjecture for quadratic modules". Recent results on quadratic modules ([1]) are precisely what is needed for the solution of the network theory problem. The vital concept of feedback is considered in Section 5 together with its relation to the problem of pole placement (stabilization) or coefficient assignment. Equivalence under feedback involves the action of a group which is not reductive. Nonetheless, a moduli space can be constructed and its invariants calculated. For controllable systems, the result can be found in [7], [28], [32] and [42]. The general case is completely analyzed in [19]. It should also be possible to extend Mumford's geometric invariant theory to groups of feedback type. A critical portion of the feedback invariants is the Kronecker set of indices $(\kappa_1, \ldots, \kappa_n)$ where the κ_i are non-negative integers. This set was used to define an ordering for systems by Rosenbrock ([42]). It turns out that this ordering is precisely the Harder-Narasimhan ordering ([45]) and we conclude with an indication of the use of this complex of ideas in system theoretic problems. We hope to indicate throughout the richness of the relationship between algebraic geometry and system theory both in terms of the application of algebraic geometry to system theory and in terms of the generation of problems in algebraic geometry from the practical considerations of system theory.

2. Routh-Hurwitz Theory

We begin by considering two questions, which were posed and solved in the 19^{th} century by Cauchy, Hankel, Hermite, Hurwitz, Maxwell, Routh and others, and which now form part of the folklore of linear system theory.

Question 1. When is a proper (i.e., vanishing at ∞) meromorphic function f on $\mathbb{C}$ rational?

Question 2. If such an f is real, when do its poles lie in the left-half plane?

In fact, we can ask the same questions for proper, matrix-valued meromorphic functions and, as a motivating example, consider the resolvent or transfer function, $f(s) = (sI-F)^{-1}$, of a differential equation, $(\frac{d}{dt} - F)x = 0$. This is still more interesting in the control system case (see Section 3), but serves to explain Maxwell's interest ([38]) in such questions. Question 1 can be thought of in terms of an algebro-geometric classification of differential equations, whereas Question 2 characterizes asymptotic stability.

One approach to Question 1, due to Hankel ([20]), is to form the matrix, $H_f = (h_{i+j-1})$, where h_ℓ is given by

$$f(s) = \sum_{\ell=1}^{\infty} h_\ell s^{-\ell}. \tag{2.1}$$

Hankel's Theorem then asserts: f is rational if, and only if, rank $H_f = n < \infty$, as one can plainly see. In this case, f induces a holomorphic map

$$f\colon \mathbb{C}\mathbb{P}^1 \to \mathbb{C}\mathbb{P}^1 \tag{2.2}$$

whose degree is given by

$$\deg_{\mathbb{C}} f = \operatorname{rank}(H_f). \tag{2.3}$$

Now, in case f is real, we can restrict

$$f\colon \mathbb{R}\mathbb{P}^1 \to \mathbb{R}\mathbb{P}^1 \tag{2.4}$$

and, since $f \mapsto H_f$ is bijective, we can ask for $\deg_{\mathbb{R}} f$ in terms of H_f. In this way, we obtain the beautiful result which underlies the Routh-Hurwitz conditions:

<u>Theorem 2.5</u> (Hermite-Hurwitz) $\deg_{\mathbb{R}} f = \operatorname{signature}(H_f)$.

This was discovered by Hermite in 1856 ([29]) in case f has distinct poles, and later extended by Hurwitz. Of course, $\deg_{\mathbb{R}} f$ was expressed in a different way:

<u>Definition 2.6</u> (Cauchy) The local index of a real, rational f at a real pole x_0 is $+1$ if f changes from $+\infty$ to $-\infty$, -1 if the alternate change occurs, and 0 if f does not change sign while passing though x_0. The index of f, $C(f)$, is the sum of the local indices.

$C(f)$, which is clearly the winding number of the map in (2.4), was defined by Cauchy in [15]. In part I of [15], he uses the

Cauchy index to compute the number of real roots of a real polynomial (generalizing, among other things, Descartes' rule of signs), the number of negative real roots, and related questions. In part II, he uses the Cauchy index to define and evaluate the local index of a non-degenerate plane vector field at an equilibrium point. Of course, this latter application was generalized by Kronecker, for $n \geq 2$, and others. On the other hand, the computation of the number of negative real zeroes of a polynomial is a bit harder than Question 2 and this computation was, in fact, the starting point for Hurwitz in [30]. Following [30], we assume for the moment that f has no poles on the imaginary axis. Thus, if g is the monic denominator of f, having no roots in common with the associated numerator h, computing the change in $\arg g(-is)$, s between $+\infty$ and $-\infty$, is equivalent to computing $p - q$, the number of zeroes in the left-half plane minus the number of zeroes in the right-half plane. This is, by trigonometry, the Cauchy index of the rational function v/u where

$$cg(-is) = (u+iv)(s),$$

and c is a complex constant rendering v/u proper. By the Hermite-Hurwitz Theorem,

$$p - q = \operatorname{sign} H_{u/v}. \tag{2.7}$$

Thus, by the Jacobi-Frobenius algorithm, one obtains numeric criteria (in terms of the coefficients of g!) for Question 2; i.e., for g to be a Hurwitz polynomial. Explicitly, this is positive-definiteness of a quadratic form, and one should stress that it is a known fact in the folklore that $H_{u/v}$ is intimately related to a Liapunov function for $g(\frac{d}{dt})x = 0$. In fact, this is very much the spirit of Hermite's treatment.

We close the section by sketching a moduli-theoretic proof of Theorem 2.5, due to R. W. Brockett ([2]). With the notation as above, consider

$$\mathrm{Rat}(n) = \{(g,h): g/h \text{ is proper and } (g,h) = 1 \text{ in } \mathbb{R}[s]\}.$$

As the colocus of the resultant, Rat(n) is Zariski open in $\mathbb{R}^{2n}$, the Euclidean topology being the same as the compact-open topology on the maps (2.4). In particular, the Cauchy index $p - q$ is constant on Euclidean components of Rat(n) and there are at least $n + 1$ of these. By continuity, $\mathrm{sign}(H_f)$ is constant on components of Rat(n), in light of (2.3). Thus, already, a general position argument reduces Theorem 2.5 to the result obtained by Hermite. Indeed, if Rat(p,q) is the submanifold on which the Cauchy index is $p - q$, then, by a somewhat tedious but intuitive canonical form argument based on divisors, Brockett proved

<u>Theorem 2.8</u>. Rat(p,q) is connected.

It is entirely trivial to evaluate the induced bijection between values of the Cauchy index and the signature of H_f. This yields Theorem 2.5 as well as some insight into the global structure of Rat(n) (i.e., of scalar input-output linear systems). It is very natural then to study the topology of the submanifolds, $Rat(p,q) \subset \Omega^1_{p-q}(S^1)$. For example, based on unpublished work of R. Brockett and G. Segal, one can show that each $Rat(p,q)$ can be given the structure of a Stein manifold and, in particular, deduce the vanishing of its higher cohomology groups. Their work also shows that $Rat(p,q)$ is simply connected only in case either $p = n$, or $q = n$, in which case $Rat(n,0) \simeq Rat(0,n) \simeq \mathbb{R}^{2n}$. It is also known that $Rat(n-1,1) \simeq Rat(1,n-1) \simeq S^1 \times \mathbb{R}^{2n-1}$ (see [2]). In another direction ([13]), from all of the above, we can construct a vector bundle V_+ on $Rat(p,q)$, obtained by assigning the positive eigenspace of H_f to $f \in Rat(p,q)$. This is, in general, non-trivial. For example, on $Rat(1,1)$, V_+ induces a classifying map

$$T_+ : Rat(1,1) \to \mathbb{RP}^L$$

which is not homotopic to a constant map. Indeed ([2]), under the isomorphism $Rat(1,1) \simeq S^1 \times \mathbb{R}^3$, T_+ is just projection on the first factor. In section four, we will interpret these obstructions in a system-theoretic context.

3. Linear Systems

The class of systems which we consider is the class of finite-dimensional, linear time-invariant multivariable systems. Such systems can be characterized via an external intrinsic description using either the transfer matrix or an input-output mapping. Associated with each description is an internal representation and an appropriate notion of equivalence which is defined by the action of an algebraic group. The critical link between these descriptions is provided by (a generalization of) Hankel's Theorem on the rationality of proper matrix valued "meromorphic" functions.

Now let $\underline{R}$ be a Noetherian integral domain which is integrally closed in its quotient field $\underline{K}$ and let $\underline{R}[s]$ be a polynomial ring in s over $\underline{R}$. We call an element $n(s)/d(s)$ of $\underline{K}(s)$ proper if degree $n(s) <$ degree $d(s)$. Let $M_{p,m}(\cdot)$ denote the set of $p \times m$ matrices with entries in $\cdot$. Then we have:

Definition 3.1. Let $\Sigma_{p,m} = \{T(s) \in M_{p,m}(\underline{K}(s)) |$ the entries $t_{ij}(s)$ of $T(s)$ are proper$\}$. Elements of $\Sigma_{p,m}$ are called proper transfer matrices.

Definition 3.2. Let $T(s)$ be an element of $\Sigma_{p,m}$. A pair $(R(s),P(s))$ with $R(s) \in M_{p,m}(\underline{K}[s])$ and $\underline{P}(s)$ an invertible element of $M_{m,m}(\underline{K}[s])$ is called a representation of $T(s)$ if $T(s) = R(s)P^{-1}(s)$. A representation is called a realization (or minimal representation) of $T(s)$ if $R(s)$ and $P(s)$ are relatively right prime.

If $T(s) \in \Sigma_{p,m}$, then, letting $\Delta(s)$ be the (monic) least common multiple of the denominators of the entries $t_{ij}(s)$ of $T(s)$, we have $T(s) = R(s)P^{-1}(s)$ where $R(s) = T_0 + T_1 s + \cdots + T_{n-1}s^{n-1}$, $P(s) = \Delta(s)I_m$ and so, representations exist. Moreover, let $\mathcal{U}_m = \{U(s) \in M_{m,m}(\underline{K}[s])$: det $U(s)$ is a unit i.e., a non-zero element of $\underline{K}\}$. Note that $\mathcal{U}_m$ is an algebraic group which acts on pairs $(R(s),P(s))$ via right multiplication and preserves the property of being relatively right prime. We now have:

__Theorem 3.3__. Given $T(s) \in \Sigma_{p,m}$, there exist realizations of $T(s)$ and any two realizations are equivalent under the action of $\mathcal{U}_m$. (For a proof, see [18], [49].)

Let $T(s)$ be an element of $\Sigma_{p,m}$ and let $\sigma_T(s)$ be an element of $M_{p+m,m}(\underline{K}[s])$ which corresponds to $T(s)$. In other words, $\sigma_T(s)$ is an element of $M_{p+m,m}(\underline{K}[s])$ such that $\sigma_T(s) = \begin{bmatrix} R_T(s) \\ P_T(s) \end{bmatrix}$ with $R_T(s), P_T(s)$ relatively right prime and $T(s) = R_T(s)P_T^{-1}(s)$. Any such $\sigma_T(s)$ is called a __linear system with transfer matrix__ $T(s)$. If $\Sigma_{p,m}$ is viewed in this way, then $\Sigma_{p,m}$ is stable under the action of $\mathcal{U}_m$ and the orbits correspond to the transfer matrices. More precisely, if $S_{p,m} \subset M_{p+m,m}(\underline{K}[s])$ is the set of all linear systems, then $S_{p,m}$ is stable under the action of $\mathcal{U}_m$ and the transfer matrix is a complete invariant for this action. These results can be interpreted in still another way which is frequently useful in applications. Namely, let $\sigma(s)$ be an element of $S_{p,m}$ and let $\sigma_j(s)$ be the j^{th} column of $\sigma(s)$. Then $\sigma_1(s),\ldots,\sigma_m(s)$ are free over $\underline{K}[s]$ and we let M_σ be the free module with generators $\sigma_1(s),\ldots,\sigma_m(s)$. M_σ is called the __system module of__ $\sigma(s)$ and is also a complete invariant for the action of $\mathcal{U}_m$.

If $\sigma(s) = \begin{bmatrix} R_\sigma(s) \\ P_\sigma(s) \end{bmatrix}$, then n_σ = degree det $P_\sigma(s)$ is called the <u>McMillan degree of</u> $\sigma(s)$ and n_σ is also an invariant for the action of $\mathcal{U}_m$. It is rather tempting to heuristically view $(R_\sigma(s), P_\sigma(s))$ as "homogeneous coordinates" relative to multiplication by $U(s)$ in $\mathcal{U}_m$. Indeed, in the scalar case $(p = m = 1)$, this is expressed in (2.2). More precisely, $\sigma(s)$ induces a map $\underset{\sim}{T}_\sigma$ of $\mathbb{P}^1$ into $\mathrm{Grass}_{\underline{K}}(m,m+p)$ given by

$$\underset{\sim}{T}_\sigma(s) = \{(R_\sigma(s)w, P_\sigma(s)w| \ w \in \underline{K}^m\} \tag{3.4}$$

for $s \neq \infty$ and by $\underline{K}^m$ for $s = \infty$. $\underset{\sim}{T}_\sigma$ is essentially the graph of $T_\sigma(s)$ by virtue of the fact that $R_\sigma(s), P_\sigma(s)$ are relatively right prime ([27]). In addition, if $\underline{K}$ is algebraically closed, then (2.3) generalizes also to the following

$$\deg_{\underline{K}} \underset{\sim}{T}_\sigma = n_\sigma \tag{3.5}$$

which was first proved over $\mathbb{C}$ by Hermann and Martin ([27]). We shall soon see that n_σ is also the rank of a Hankel matrix.

We now turn our attention to the alternate external description of linear systems. Consider the (formal) Laurent series

$$f(s) = \sum_{\ell=1}^{\infty} L_\ell s^{-\ell} \tag{3.6}$$

where $L_\ell \in M_{p,m}(\underline{K})$. We associate with f the (generalized) Hankel matrix

$$H_f = (L_{i+j-1}) \tag{3.7}$$

and we have:

<u>Definition 3.8</u>. f is <u>admissible</u> if rank $H_f = n_f < \infty$. Let $\Sigma^f_{p,m} = \{f(s) \mid f$ is admissible$\}$. Elements of $\Sigma^f_{p,m}$ are called <u>proper input-output maps</u> and n_f is called the <u>dimension of f</u>.

<u>Definition 3.9</u>. Let $f(s)$ be an element of $\Sigma^f_{p,m}$. A triple (A,B,C) with $A \in M_{n,n}(\underline{K})$, $B \in M_{n,m}(\underline{K})$, $C \in M_{p,n}(\underline{K})$ is called a <u>representation</u> of $f(s)$ if $CA^{\ell-1}B = L_\ell$ for $\ell = 1,\ldots$. A representation is called a <u>realization</u> (or minimal representation) of $f(s)$ if $n = n_f$.

<u>Definition 3.10</u>. A representation (A,B,C) of $f(s)$ is called <u>controllable</u> if rank $\mathscr{C}(A,B,C) = \text{rank}\,[B,AB,\ldots,A^{n-1}B] = n$ and a representation (A,B,C) of $f(s)$ is called <u>observable</u> if rank $\mathscr{O}(A,B,C) = \text{rank}[C',A'C',\ldots,(A')^{n-1}C'] = n$.

If $f(s)$ is a proper input-output map, then it is well-known that representations and realizations exist ([33]). Moreover, if (A,B,C) is a representation of $f(s)$, then $C(sI-A)^{-1}B = T_f(s)$ is an element of $\Sigma_{p,m}$ and (A,B,C) can be called a <u>state-space representation</u> of $T_f(s)$. Similarly, if (A,B,C) is a realization of $f(s)$, then $C(sI-A)^{-1}B = T_f(s)$ is an element of $\Sigma_{p,m}$ and (A,B,C) is called a state-space realization of $T_f(s)$. Thus, there is a natural mapping of $\Sigma^f_{p,m}$ into $\Sigma_{p,m}$ and it can be shown that this mapping is bijective by virtue of Hankel's Theorem ([20]).

Now, if (A,B,C) is a triple with $A \in M_{n,n}(\underline{K})$, $B \in M_{n,m}(\underline{K})$ $C \in M_{p,m}(\underline{K})$, then $GL(n,\underline{K})$ acts on (A,B,C) in the following way: $(A,B,C) \to (gAg^{-1}, gB, Cg^{-1})$, $g \in GL(n,\underline{K})$. It is clear that controllability and observability are preserved under this action. We now have:

<u>Theorem 3.11</u>. Given $f(s) \in \Sigma^f_{p,m}$ (or, equivalently, $T_f(s) \in \Sigma_{p,m}$), there exist realizations of $f(s)$ (state-space realizations of $T_f(s)$) and any two realizations are equivalent under the action of $GL(n_f,\underline{K})$. Moreover, any realization of $f(s)$ is both controllable and observable. Finally, if (A,B,C) and (A_1,B_1,C_1) are realizations of $f(s)$, then the $g \in GL(n_f,\underline{K})$ such that $(gAg^{-1}, gB, Cg^{-1}) = (A_1,B_1,C_1)$ is unique.

(For a proof, see [3], [17]).

The final part of Theorem 3.11 is often referred to as the state-space isomorphism theorem.

So, let $f(s) \in \Sigma^f_{p,m}$ and let (A,B,C) be an element of $\mathbb{A}^{n_f^2+n_f(m+p)}(\underline{K}) = \mathbb{A}^{n^2+n(m+p)}$ which corresponds to $f(s)$. In other words, (A,B,C) is an element of $\mathbb{A}^{n^2+n(m+p)}$ such that (A,B,C) is both controllable and observable and $T_f(s) = C(sI-A)^{-1}B$ and $f(s) = \sum_{\ell=1}^{\infty} (CA^{\ell-1}B)s^{-\ell}$. Any such triple (A,B,C) is called a <u>linear system with input-output map</u> $f(s)$. Let $S^n_{p,m}$ be the set of all linear systems of dimension n. Then $S^n_{p,m}$ is stable under the action of $GL(n,\underline{K})$ and the orbits correspond to the proper input-output maps of dimension n. In the next section, we shall characterize the moduli space of $S^n_{p,m}$ under this action. Finally, we remark that the dimension of $f(s)$ is precisely the McMillan degree of $T_f(s)$.

So far, we have considered notions of equivalence based on classical groups such as $\mathscr{U}_m$ and $GL(n,\underline{K})$. However, in Section 5, we will examine the critical concept of feedback equivalence. This notion requires, amongst other things, an additional set of invariants known as the Kronecker indices ([14], [32]) which we now introduce in two ways.

If $P(s) \varepsilon M_{m,m}(\underline{K}[s])$, then let $\kappa_j(P) = \max \text{degree}\{p_{ij}(s)$ $i = 1,\ldots,m\}$ be the j^{th} column degree of $P(s)$ so that $P(s)$ can be written in the form

$$P(s) = \Delta_c(P)\text{diag}[s^{\kappa_1},\ldots,s^{\kappa_n}] + P_1(s) \qquad (3.12)$$

where $\kappa_j(P_1) < \kappa_j(P)$ and $\Delta_c(P) \varepsilon M_{m,m}(\underline{K})$. $P(s)$ is called column proper if $\Delta_c(P) \varepsilon GL(m,\underline{K})$. If $\sigma = \sigma(s) = \begin{bmatrix} R_\sigma(s) \\ \\ P_\sigma(s) \end{bmatrix}$ is an is an element of $S_{p,m}$, then it is well-known that there is a $U(s) \varepsilon \mathscr{U}_m$ such that $P_\sigma(s)U(s)$ is column proper ([ ]). We now have:

Definition 3.13. Let σ be an element of $S_{p,m}$ and let $P(s)$ be a column proper element of $M_{m,m}(\underline{K}[s])$ such that $P(s) = P_\sigma(s)U(s)$ for some $U(s) \varepsilon \mathscr{U}_m$. Then the set of integers $\kappa_\sigma = \{\kappa_1(P),\ldots,\kappa_m(P)\}$ is called the Kronecker set of σ.

Theorem 3.14. Let σ be an element of $S_{p,m}$. Then (i) κ_σ is well-defined, and (ii) if $\tau = \sigma U$ for some $U \varepsilon \mathscr{U}_m$, then

$\kappa_\sigma = \kappa_\tau$ (as sets). (i.e., κ_σ is an invariant for the action of $\mathcal{U}_m$ on $S_{p,m}$).

Alternatively, in state-space form, let (A,B,C) be controllable so that $\operatorname{rank}[B,AB,\ldots,A^{n-1}B] = n$. Then it is possible to lexicographically order the first n linearly independent columns of $[B,AB,\ldots,A^{n-1}B]$ as follows: $b_1, Ab_1, \ldots, A^{\kappa_1 - 1} b_1$, $b_2, \ldots, A^{\kappa_2 - 1} b_2, \ldots, A^{\kappa_m - 1} b_m$ where $b_1, \ldots, b_m$ are the columns of B. The set of integers $\kappa = (\kappa_1, \ldots, \kappa_m)$ is called the Kronecker set of (A,B,C). It can be shown that κ is an invariant under the action of $GL(n,\underline{K})$ ([7]). For example, if we consider the pencil $[A-sI,B]$, then equivalence under $GL(n,\underline{K})$ induces a strict equivalence (in the classical sense of Kronecker [ ]) of pencils and the integers $(\kappa_1, \ldots, \kappa_m)$ are a complete invariant for strict equivalence of pencils. Explicitly, following Kronecker, the κ_i may be computed by the process: choose $\phi_1(s) \in \operatorname{Ker}[A-sI,B] \otimes \underline{K}$ of minimal degree κ_1 in s_j then choose $\phi_2(s) \in \operatorname{Ker}[A-sI,B] \otimes \underline{K}$, independent of $\phi_1(s)$, and of minimal degree κ_2 among all such $\phi(s)$, etc. (cf. [10]). We also have:

Theorem 3.15. Let (A,B,C) be a state-space realization of T(s) and let $\sigma_T(s)$ be a realization of T(s). Then the Kronecker sets are the same.

(For a proof, see [18] and [19].)

We note that, for ease of exposition, we have worked over the quotient field $\underline{K}$ of $\underline{R}$. However, if the data were defined over $\underline{R}$, then the realization problem is essentially a question of rationality which accounts for the assumption of integral closure. Thus, the results presented here hold with data in $\underline{R}$ (see, [43], [44], [47]).

4. Classical Groups of Symmetry and Moduli for Linear Systems

There are three main topics covered here. The first is the construction, in external and in internal terms, of the moduli space for linear systems and a proof of the theorem on "realization with parameters", which is based on Zariski's Main Theorem. The second is a brief survey of biregular and topological invariants of the moduli space, with emphasis on the impact these invariants have on system theory, as in Section 2. We close by considering moduli for systems having richer external symmetries, such as Hamiltonian systems or systems describing symmetric electrical networks. Here we see a more arithmetic side come to the foreground, and in fact we sketch the equivalence between an old problem in network synthesis and what might be called the "Serre conjecture for quadratic modules". In particular, recent results on quadratic modules ([1]) turn out to be sufficient for the classical formulation of this problem.

Now, over $\mathbb{R}$ or $\mathbb{C}$, it is natural to view the set of transfer functions with m inputs, p outputs, and fixed McMillan degree n as a (complete) metric space. Indeed, this topology ([13]) is simply the compact-open topology on transfer functions, regarded as maps

$$\underset{\sim}{T}_{\sigma}: \mathbb{P}^1 \to \mathrm{Grass}(m,m+p),$$

as in (3.4). Although such a description is in fact useful in certain contexts, it is not entirely clear that this space is an algebraic variety, nor is it even clear what its dimension should be. For this reason, the first problem we consider is to construct, as explicitly as possible, a moduli space for the action of $GL(n,\underline{k})$ on $S^n_{p,m}$. In fact, since it is rather important as a classifying space for families of linear systems, it is rather crucial that we construct a fine moduli space. This is done in several stages, it turns out (by universality) to be sufficient to construct a geometric quotient for the action of $GL(n,\underline{k})$ on controllable pairs (A,B) in $\mathbb{A}^{n^2+nm}$. This is done within the context of geometric invariant theory, the rather remarkable result being that the properly stable points in $\mathbb{A}^{n^2+nm}$, relative to the standard character linearization, is precisely the space of controllable pairs!. We start birationally.*

First of all, the corresponding space $\mathbb{H}$ of Hankel's is clearly a quasi-affine variety, contained in $\mathbb{A}^N$, for $N = mp(n+1)$. This was shown to be non-singular by J.M.C. Clark ([16]) and, in addition, his proof shows that $\mathbb{H}$ is a special variety, in the sense of Chevalley. In particular, $\mathbb{H}$ is rational and one can easily write down generators for the function field $\underline{K}_{\mathbb{H}}$. On the other hand, if $\underline{R} = \underline{k}[A,B,C]$, $\underline{K}$ the field of fractions of $\underline{R}$, then Falb has shown in [18]:

<u>Proposition 4.1</u>. $\underline{K}_{\mathbb{H}} = \underline{K}^{GL(n,\underline{k})}$.

This follows from the statement that the ring of invariant regular functions defined on the Zariski open subspace of semi-simple A is generated by the entries of the Hankel. This is proved,

*The birational situation using these ideas was sketched in [18] and the entire theory fully developed in [10].

for B and C vectors, by Gurevitch - as an illustration of the symbolic method! It is proved in general in [18].

The biregular situation involves a bit more care - this is due to several people (e.g. [14], [26]) and our treatment follows ([14]), relying on [40]. It can be shown that, if $\tilde{\Sigma}_0$ is the Zariski open subspace of $\mathbb{A}^{n^2+nm}$ of controllable pairs and if R_0 is the ring of regular functions on $\tilde{\Sigma}_0$, then in contrast to Proposition 5.1, $R_0^{GL(n,\underline{k})}$ does not separate orbits in $\tilde{\Sigma}_0$, unless $m = 1$. In fact, if $\underline{k}$ is algebraically closed,

$$R_0^{GL(n,\underline{k})} = \underline{k}[c_i], \tag{4.2}$$

where (c_i) denote the characteristic coefficients of A. In particular, although Theorem 3.9 implies that the orbits in $\tilde{\Sigma}_0$ are closed, an affine quotient does not exist, unless $m = 1$. However, we do have:

__Theorem 4.3__. $\tilde{\Sigma}_0 \to \Sigma_0$ is a universal geometric quotient, with Σ_0 a quasiprojective, non-singular, irreducible variety of dimension nm.

__Sketch of Proof__. Following [40], since (4.2) implies that the set of properly stable points for the trivial character ($\mathrm{Pic}(\tilde{\Sigma}_0) = (0)$!) is empty, we compute the properly stable points for the relative invariants of weight 1. Here we find a surprisingly pleasant result:

Proposition 4.4. The set of properly stable points, relative to $\det(\cdot)$, in $\mathbb{A}^{n^2+nm}$ is exactly $\tilde{\Sigma}_0$.

This is very much related to the fact that $\tilde{\Sigma}_0$ is the principal orbit type for the action. In the scalar input case, this can be proved by linear algebra, one direction being already implied by the proof of the state-space isomorphism theorem. For the other half, to say (A,B) is in the principal orbit type is to say, by differentiating the action,

$$n^2 = \dim \mathcal{O}_{(A,B)} = \dim \mathcal{O}_A + \dim Z(A)B, \tag{4.5}$$

where $Z(A)$ is the centralizer of A and $\mathcal{O}_\cdot$ denotes the orbit of $\cdot$. Now, (4.5) can be refined since

$$\dim \mathcal{O}_A \leq n^2 - n \quad \text{and} \quad \dim Z(A)B \leq n \tag{4.6}$$

holds for any A. Thus, $\dim \mathcal{O}_A = n^2 - n$ is maximal and A therefore possesses a cyclic vector. By Frobenius's Theorem, $Z(A) = k[A]$ and thus

$$\dim Z(A)B = \dim k[A]B = \operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n,$$

which was to be shown. The multivariable case also follows from linear algebra (see [14]).

Assuming Proposition 4.4, Theorem 4.3 follows from Mumford's Theorem ([40], Theorem 1.19).

By universality, a geometric quotient for the twisted action of $GL(n,\underline{k})$ on $\tilde{\Sigma}_p = \{(A,B,C)\colon (A,B)$ controllable$\}$ exists and shares many of the same properties. In fact, $\Sigma_p \to \Sigma_0$ is an algebraic vector bundle with base Σ_0 and fiber k^{np}. Since the quotient, $\tilde{\Sigma}_p \to \Sigma_p$ is geometric, we also obtain a quotient $\Sigma^n_{p,m}$ for the Zariski open subspace $S^n_{p,m} \subset \tilde{\Sigma}_p$.

<u>Theorem 4.7</u>. A fine moduli space for state-space equivalence in $S^n_{p,m}$ exists. Moreover, $\Sigma^n_{p,m}$ is a rational, smooth, irreducible quasiprojective variety of dimension $n(m+p)$.

The construction of a universal family of systems on Σ_0 was carried out in [26] and extended to $\Sigma^n_{p,m}$ in [10]. Thus, any map $f\colon X \to \Sigma^n_{p,m}$ induces a linear system defined over the coordinate ring R of X. This, of course, extends to the study of differentiable or continuous families as well.

<u>Theorem 4.8</u>. If $H(x)$ is a regular Hankel matrix, of locally constant rank on X, then $H(x)$ is realizable over R, with finitely generated, projective state module.

<u>Proof</u>. After all of the above, including the existence of a universal family, i.e. a vector bundle Q - the state bundle, an endomorphism $\mathscr{A}$ of Q, and sections and co-sections $\mathscr{B}_i$, $\mathscr{C}_j$, resp., this amounts to proving that realization is, in fact, algebraic in its parameters (see [25], [14]). That is, if η is the natural map

$$\eta \colon \Sigma^n_{p,m} \to \mathbb{H} \qquad (4.9)$$

$$[A,B,C] \to (CA^{i+j-1}B)$$

we claim η is biregular. By Theorem 3.9, η is bijective and by Proposition 4.1 η is birational. By Clark's Theorem, $\mathbb{H}$ is normal and thus η is biregular by Zariski's Main Theorem. Finally, the assumption on $H(x)$ is simply that H define a map to $\mathbb{H}$.

<u>Example 4.10</u>. Given $(h_{ij}) \in \ell^1(\mathbb{Z})^{(n+1)}$ of rank n, we seek a realization $(A(z),B(z),C(z))$ summable in $z \in \mathbb{Z}$. If we ask that the dual system be minimal too, the following sufficient condition (see [10], [46]) becomes necessary.

(T) the closed linear span of $\det(h_{ij})$ and its translates is $\ell^1(\mathbb{Z})$.

Indeed, Fourier transforming and using the Tauberian Theorem, we obtain a continuous map

$$\widehat{(h_{ij})} \colon S^1 \to \mathbb{H} \qquad (4.11)$$

and, thus, a linear system $(\hat{A},\hat{B},\hat{C})$ defined over $\widehat{\ell^1(\mathbb{Z})}$, since η^{-1} is analytic. Now, the state module is trivial over $C(S^1)$, for topological reasons, and the Docquier-Grauert Theorem asserts that the corresponding bundle can even be trivialized over $\widehat{\ell^1(\mathbb{Z})}$! In particular, we obtain a realization, in matrix form, by inverting the Fourier transform. Recent work by M. Hazewinkel on constructing partial compactifications for $\Sigma^n_{p,m}$, suggests that we may be able to drop the hypothesis on the Hankel.

We stress the fact that the computation made above actually depends on the topology of S^1; for example, it can be shown that there exists $\mathbb{P}^N \subset \Sigma^n_{p,m}$, of codimension $n(p+1)$, over which the universal family is non-trivial. This is, of course, very much related to the existence of algebraic (or continuous) canonical forms - a problem which has received a great deal of attention.

Now, Mumford's Theorem yields an actual imbedding of Σ_0 in $\mathbb{P}^M$; i.e. a very ample line bundle L on Σ_0. L also arises in another way, viz. $L = \Lambda^n \Sigma_1$, as one can see by checking cocycles (c.f. [14], Section 4), and thus L is associated to the principal $GL(n,\underline{k})$-bundle, $\tilde{\Sigma}_0 \to \Sigma_0$. Furthermore, L cannot be trivial if $m > 1$, by (4.2). If $m = 1$, $\Sigma_0 \simeq \mathbb{A}^n$ by the existence of the rational canonical form. The more precise results for $m > 1$ are:

<u>Theorem 4.12</u>. If k is algebraically closed, $\mathrm{Pic}(\Sigma_0) = (L) \simeq \mathbb{Z}$ and there exist no algebraic canonical forms. If $k = \mathbb{C}$, then the Betti numbers β_{2i} do not vanish, for $i = 0,\ldots,nm - n$ and continuous canonical forms do not exist.

<u>Sketch of Proof</u>. Since $\tilde{\Sigma}_0 \to \Sigma_0$ is an algebraic principal $GL(n,k)$-bundle, one can conjugate $\mathrm{Pic}(\Sigma_0)$ by descent ([40], p. 32):

$$\mathrm{Pic}^{GL(n,\underline{k})}(\tilde{\Sigma}_0) \simeq \mathrm{Pic}(\Sigma_0).$$

Since $\mathrm{Pic}(\tilde{\Sigma}_0) = (0)$, we have a surjection (by the rigidity lemma),

$$\widehat{GL(n,k)} \to \mathrm{Pic}^{GL(n,k)}(\tilde{\Sigma}_0)$$

which is injective by (4.2). In terms of group cohomology, this is just the map,

$$\widehat{GL(n,\underline{k})} \to \underline{H}^1(GL(n,\underline{k}),k^*) \to H^1(GL(n,\underline{k}),\mathcal{O}^*_{\Sigma_0}) = \mathrm{Pic}^{GL(n,\underline{k})}(\Sigma_0),$$

induced by the inclusion of $GL(n,\underline{k})$-modules, $k^* \to \mathcal{O}^*_{\Sigma_0}$. If $m = 1$, the unit $\det(B,AB,\ldots,A^{n-1}B)$ is a coboundary of the character $\det(\cdot)^\ell$, regarded as a cocycle with value in $\mathcal{O}^*_{\Sigma_0}$.

As for the topological obstructions, notice that (4.2) asserts that $\mathbb{A}^n$ is the categorical quotient for the action, in the category of affine varieties. In particular, consider the natural foliation

$$\Sigma_0 \to \mathbb{A}^n.$$

In the codimension 1 case, $n = 1$ and thus $\Sigma_0 = \mathbb{A}^1 \times \mathbb{P}^{m-1}$ while $\tilde{\Sigma}_0 \to \Sigma_0$ is the pullback of the universal fibration. This is quite general, on a Zariski open subspace of $\mathbb{A}^n$, the leaves are co-*dimension n* projective spaces, $\mathbb{P}^N$! Over $\mathbb{C}$, these can be imbedded non-trivially, giving a proof of the second statement. Q.E.D.

Over $\mathbb{R}$, M. Hazewinkel ([24]) has shown that there exist no continuous canonical forms, by a clever imbedding of S^1 into Σ_0. It is possible to prove non-vanishing of higher homotopy groups in the proper ranges over $\mathbb{R}$ and $\mathbb{C}$, by making some crude computations along the lines in [11], and applying Bott Periodicity. These computations also show that the natural foliation on Σ_0 is not, in general, a product of $\mathbb{A}^n$ with a typical leaf. Of course, the situation remains the same for Σ_p, but $\Sigma^n_{p,m}$ is a bit more elusive. However, the statements about canonical forms still hold, answering in the negative - by topological and geometric means - a question which many engineers had assumed to be evident. This global form of the question was originally raised by R. E. Kalman [31].

For the remainder of the section, we consider the real field $\mathbb{R}$, and discuss an application, following [13], of a theorem of Harder to a problem in network synthesis, posed by Youla in [53]. It should be stressed that, even a decade ago, this problem was known to be algebro-geometric in nature and, in fact, the partial results obtained by Koga ([36]) were based on his use of the Riemann-Roch theorem for curves to analyze coprime factorizations of transfer functions in two variables.

Now, a result of Glover ([21]) asserts that if $m > 1$ or $p > 1$, the space $\mathbb{H}$ is connected in contrast to Theorem 2.7. This is due largely to the fact that, even in case $m = p$, the Hankel is not necessarily symmetric and one has more room to move in. However, symmetric Hankels or, equivalently, symmetric transfer functions are important in network theory (see [6] , [37], [54]). This symmetry is displayed in state space-form as:

$$I_{p,q}A = A'I_{p,q}$$
$$I_{p,q}B = C', \tag{4.13}$$

where $I_{p,q}$ is the standard form of signature $p - q$. Here $p - q$ is the signature of the Hankel and has the same physical significance

([4]) as the signature of the indefinite metric in the Brayton-Moser equations. In fact, the state-space isomorphism theorem applies here to give the group of symmetrices $\mathcal{O}(p,q) \subset GL(n,\mathbb{R})$ (see [54]). Moduli for these linear dynamical systems have been studied in [13] where, in particular, Theorem 2.7 is shown to hold. The invariant playing the role of the Cauchy index is the Maslov index where the real Grassmann in (3.4) is replaced by a Lagrangian Grassmann.

Now, the problem alluded to above is just a question of realization with parameters, subject to the symmetry constraints. That is, given a symmetric transfer function involving delays - which is simply a transfer function over $R[x_1,\ldots,x_N]$ ([34], [35]) - find a symmetric realization. By the validity of the Serre conjecture, the state module can be assumed free and the corresponding question is equivalent ([13]) to:

> given a symmetric Hankel with constant rank, defined over $R = \mathbb{R}[x]$, find a symmetric realization (4.13) over R.

Since, over $\mathbb{R}$, this depends only on Sylvester's Theorem, we can invoke the following to finish off the problem.

<u>Theorem</u> (Harder [1]). Any non-degenerate symmetric bilinear form defined over $k[x]$ is equivalent to a form defined over k.

We close by remarking that the general theory of quadratic modules arises in several instances, especially in connection with families of Hamiltonian systems. Here, again, over $\mathbb{R}$ we have a Hamiltonian realization, with symmetry group $Sp(n,\mathbb{R})$ and thus the moduli problem is already "in canonical form".

5. Stabilization and Feedback Groups

To be sure, one of the most important results in deterministic control theory is the pole-placement theorem, Theorem 5.2, which is concerned with stabilization of linear systems by means of "state feedback" - a topic very much related to the material developed in Section 2. It is fair to add, however, that a much deeper understanding of the whole feedback question comes only with the interpretation of feedback substitutions as part of a group action on $\mathbb{A}^N$, $N = n^2 + nm + np$. Explicitly, for (n,m) fixed we define $\mathcal{F} = \mathcal{F}(n,m)$ as the semi-direct product of 3 groups, generated by the actions on triples (A,B,C):

(i) $GL(n,\underline{k})$ acting via change of basis in the state space,

(ii) $GL(m,\underline{k})$ acting via change of basis in the input space,

(iii) $\mathrm{Hom}(\underline{k}^n,\underline{k}^m)$ acting via $(A,B) \mapsto (A+BF,B)$.

This action is more complicated than the action in (i), for both mathematical and control-theoretic reasons. That is, $\mathcal{F}$ is a non-reductive extension of $GL(n,\underline{k})$ and, although $\mathcal{F}$ preserves con- controllability, and hence acts on $\tilde{\Sigma}_p$, $\mathcal{F}$ does not leave the space of linear systems, $S^n_{p,m}$, invariant. Thus, we begin this section by studying the action

$$\mathcal{F} \times \tilde{\Sigma}_0 \to \tilde{\Sigma}_0 \tag{5.1}$$

and extending to the case $p > 0$. This extension is possible, in the sense of constructing moduli, since each of the isotropy subgroups arising in (5.1) is the semidirect product of a reductive group with the unipotent radical. We reproduce the theorem of

Hermann and Martin ([28]) which states that the classical invariants for (5.1) - which are arithmetic, (being the Kronecker set introduced in Section 3) - arise as the Birkhoff-Grothendieck invariants of a bundle on $\mathbb{P}^1$. We extend these considerations to families of controllable pairs and, following Harder, Narasimhan and Shatz ([45]), introduce what might be called the "ubiquitous ordering" on the set of partitions of an integer n. Indeed, this ordering is well-known in system theory, having been introduced by Rosenbrock ([42]) in his study of the generalized pole-placement theorem, and we briefly indicate (based on joint work with C. Martin) how this circle of ideas yields some powerful new tools in the analysis of state feedback.

First of all, recall that the poles of a transfer function $T(s)$ are precisely the roots of χ_A - the characteristic polynomial of A, where (A,B,C) is a linear system realizing $T(s)$. Now, if $F \in \mathrm{Hom}(R^n, R^m)$, ($R$ a commutative ring with identity) we may regard the action (iii) of F as feeding back the state of the system as a component of the input. In this setting, we ask (over $\mathbb{R}$) whether there exists an F such that χ_{A+BF} is a Hurwitz polynomial. More generally, over an arbitrary field k we consider ([51]):

Theorem 5.2. Any monic $p(s)$ can be expressed as χ_{A+BF} if, and only if, (A,B) is controllable.

In the scalar-input case, this result was classically derived from the existence of the rational canonical form, and thus holds for any ring R. However, for families of systems with $m > 1$, technicalities arising from the non-existence of canonical forms, Theorem 4.12, are non-trivial. In fact, a counterexample for the general situation has recently been announced by Bumby and Sontag ([9]). In general, our knowledge for arbitrary coefficient domains is very scant and is complete only under strong assumptions on the ring and the type of monic polynomial considered ([39]) or on the type of system ([12]).

Nevertheless, we can still glean some insight from the scalar input case. In terms of the action (5.1), which is transitive in this case, we fix (A,B) and consider

$$\tilde{\alpha}_{(A,B)}: \mathscr{F} \to R^n, \tag{5.3}$$

$$\tilde{\alpha}_{(A,B)}(\bar{g},s,F) = (c_i(\bar{g}(A+BF)\bar{g}^{-1})),$$

(where $c_i(\cdot)$ is the i^{th} characteristic coefficient). Following E. Kamen, we note that the image of $\tilde{\alpha}_{(A,B)}$ is the translation, by the vector χ_A, of a linear subspace, so that if we reduce modulo (χ_A), and supress $\bar{g}$ and s, the corresponding map

$$\alpha_{(A,B)}: \mathrm{Hom}(R^n,R) \to R^n \tag{5.4}$$

is linear! If $\mathscr{L}^{(A,B)} = [B,AB,\ldots,A^{n-1}B]$ is the controllability operator, then the relationship between $\mathscr{L}$ and α, i.e. Theorem 5.2, is brought out by the beautiful identity due to B.F. Wyman ([52]):

$$\mathrm{coker}\ \alpha \simeq \mathrm{coker}(\mathscr{L}^*) \tag{5.5}$$

If the pair (A,B) is controllable over the fraction field of R, the right hand side can be computed in terms of $\mathscr{L}$, i.e.:

$$\mathrm{Ext}\,^1_R(\mathrm{coker}\ \mathscr{L},R) \simeq \mathrm{coker}\ \alpha,\ \mathrm{Ext}\,^1_R(\mathrm{coker}\ \alpha,R) \simeq \mathrm{coker}\ \mathscr{L}. \tag{5.6}$$

In particular, if R is a field and if A is semi-simple, (5.5) contains the folklore result that one cannot change the uncontrollable modes by state feedback. That is, if A is diagonal, then any eigenvalue λ_i for which b_i is zero must persist in χ_{A+BF}.

In the case, $m > 1$, if k is algebraically closed and of characteristic 0, we can replace α by its Jacobian and thus obtain the corresponding generic result from the dominant morphism theorem. The computation of the Jacobian, over $\mathbb{C}$, is due to Hermann and Martin ([27]) and relies on a differential analogue of Proposition 4.4.

Thus far, the only invariants which we have attached to a linear system have been topological, viz. the degree of the associated transfer function,

$$\underset{\sim}{T}\colon \mathbb{P}^1_k \to \mathrm{Grass}_k(m,m+p). \tag{5.7}$$

For $k = \mathbb{R}$, this defines the Cauchy index if $m = p = 1$ or else the McMillan degree mod (2). For $k = C$, this defines the McMillan degree. However, $\underset{\sim}{T}$ is an algebraic map, and as such determines finer invariants, namely: the isomorphism class of T^*U, where U is the (algebro-geometric) universal vector bundle on $\mathrm{Grass}_k(m,m+p)$. By the well-known result of Birkhoff — Grothendieck (and probably others!), T^*U is determined by a partition of $\deg_k T$ into non-negative integers (n_i), $n_1 \geq n_2 \geq \cdots \geq n_m \geq 0$, if k is algebraically closed. (If $k = \mathbb{R}$, a theorem of Serre's still gives a decomposition of T^*U into a sum of line bundles, but these in turn are only determined by integers mod(2).) We owe to Hermann and Martin the identification of these geometric invariants ([28]):

Theorem 5.8. The Birkhoff-Grothendieck invariants of T^*U are the Kronecker set of any minimal realization.

Sketch of Proof. Since one set of invariants is defined in the frequency domain and the other via state-space techniques, the key step will involve the Laplace transform. Note that, for constant coefficient differential operators, the Laplace transform has its range in a field of transcendence degree 1 and genus 0. This theorem very clearly indicates the relevance of the genus in such questions.

The key step is due to Rosenbrock: over $\mathbb{C}$, consider the Laplace transforms of the solutions to the initial value problem,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0.$$

These are elements of the $\mathbb{C}[s]$-module, defined as the kernel of the following pencil of matrices,

$$[A-sI,B]: (\mathbb{C}^n \oplus \mathbb{C}^m) \otimes \mathbb{C}[s] \to \mathbb{C}^n \otimes \mathbb{C}[s]. \tag{5.9}$$

The theory of such pencils, particularly the problem of equivalence, is due to Weierstrass and Kronecker (see [20]) and applies remarkably well to the situation at hand. Explicitly, controllability of (A,B) is equivalent to right surjectivity of (5.9) and the notion of strict equivalence of pencils is precisely that of feedback equivalence of (A,B) with (A',B'). Indeed, strict equivalence is an equivalence of $k[s]$-module maps, with the isomorphisms independent of s. By writing the definition of strict equivalence and comparing degrees, we find that the group of strict equivalence of pencils of the form (5.9) is a matrix representation of $\mathcal{F}$! Since independence of s connotes projectivity, we follow another clue from Kronecker: we projectivize the pencil (5.9). In this setting, right surjectivity guarantees that we obtain a homogeneous bundle on $\mathbb{A}^2 - \{0\}$, and hence an m-bundle on $\mathbb{P}^1_k$. According to the Kronecker recipe, given at the end of Section 3, this bundle decomposes into a sum $\bigoplus_{i=1}^{m} \mathcal{O}(\kappa_i)$, $\sum_{i=1}^{m} \kappa_i = n$. That this bundle, $\ker[tA-sI,tB]$, on $\mathbb{P}^1_k$ is isomorphic to T^*U, follows at once from observability: we simply map $[x,u]$ to $[u,Cx]$, sending the kernel above to the graph of T. Observability is then just the statement that this map is injective.

Thus, Kronecker's classification also shows more, as noted in [32], for controllable pairs:

<u>Theorem 5.10</u>. $(A,B) \equiv (A',B') \bmod \mathcal{F}$ if, and only if, the Kronecker sets are the same.

In fact, Kronecker's proof (see [20]) is to construct a canonical form for the pencil, given the data (κ_i). This canonical form coincides with the form discovered by P. Brunovsky ([7]) by other methods. We remark, however, that as an immediate application of the construction of this form, pole-placement becomes trivial. Indeed, the invariants $n_1 \geq n_2 \geq \ldots \geq n_m \geq 0$ are much more refined, they give a "generalized pole-placement theorem".

<u>Theorem 5.11</u>. (Rosenbrock [42]). The invariant factors of A + BF can be made arbitrary, subject only to the constraints.

$$\deg \phi_1 \geq n_1, \ \deg \phi_2 \geq n_1 + n_2, \ldots, \deg \phi_\ell \geq \sum_{i=1}^{\ell} n_i .$$

Another way of interpreting T^*U as a complete feedback invariant for the action (5.1) is to count sections. Thus, if $h^0(V)$ is the dimension of $H^0(\mathbb{P}^1;V)$, we have

$$h^0(T^*U) = h^0\left(\overset{m}{\oplus}\, \mathcal{O}(n_i)\right) = \sum_{i=1}^{m} (n_i+1) = n + m, \tag{5.12}$$

On the other hand, evaluating T^*U at ∞ gives, canonically, the input space. Thus, if $\phi\colon T_1^*U \to T_2^*U$ is an isomorphism, ϕ induces an isomorphism $\phi_*\colon H^0(\mathbb{P}^1;T_1^*U) \to H^0(\mathbb{P}^1;T_2^*U)$, commuting with evaluations at ∞! Thus, ϕ_* is a triangular element of $GL(n+m,\underline{k})$, corresponding to the matrix representation of ϕ_* obtained above.

This point of view has several applications. For instance, if $(A,B) \in \tilde{\Sigma}_0$, then $\mathcal{F}_{(A,B)}$ - the isotropy subgroup for the action (5.1) - is just Aut(V), where V is the bundle defined by the kernel of [tA-sI,tB]. In particular,

$$\begin{aligned} \dim \mathcal{F}/\mathcal{F}_{(A,B)} &= \dim \mathcal{F} - h^0(V \otimes V^*) \\ &= \dim \mathcal{F} - \Big(\sum_{n_i \geq n_j} (n_i - n_j + 1) \Big), \end{aligned} \tag{5.13}$$

a formula originally derived by R. Brockett in [5], by a more explicit representation of $\mathcal{F}_{(A,B)}$ together with a heuristic argument using Young diagrams corresponding to the partitions. Based on this description, it can also be shown ([5]) that connectivity (over $\mathbb{R}$) of the orbits is determined by the Kronecker indices mod(2).

Now, $\mathcal{F}_{(A,B)}$ has also been written down explicitly by Falb and Wolovich in [19], in order to study the action of $\mathcal{F}$ (and an extension of $\mathcal{F}$) on $\tilde{\Sigma}_p$. Then, one studies, in addition to the Kronecker set of (A,B), the action of $\mathcal{F}_{(A,B)}$ on $\mathbb{A}^{np}$. In [19], it is shown that a moduli space for this action exists, using the fact that $\mathcal{F}_{(A,B)}$ is always the semi-direct product of a reductive group with its unipotent radical.

For the remainder of the section we consider only $\mathcal{F}$ acting on controllable pairs (A,B) and, as in Section 4, we begin to introduce parameters. Thus, we suppose (A,B) is an algebraic family, defined on an irreducible affine variety X. If (A,B) arises from a delay-differential system, it can be shown that we can take $X = \mathbb{A}^N$ ([10]) and that equivalence modulo $\mathcal{F}(R)$ is not just a formal equivalence ([35], [10]). In this case, one may assign geometric data to (A,B), viz.

$$V_{(A,B)} \to X \times \mathbb{P}^1_k,$$

as before, by homogenizing the Rosenbrock pencil. If $X = \mathbb{A}^N$, it may be shown ([10]), based on the alternative view presented above, that $V_{(A,B)}$ is actually a complete invariant for (A,B) under $\mathcal{F}(R)$.

This situation is an improvement, as quite a bit more is known about deformations of bundles on $\mathbb{P}^1$ than about pencils over R. For example, if the deformation is point-wise trivial, i.e. if the Birkhoff-Grothendiech invariants are constant in X, then a theorem of C. Hanna ([23]) asserts:

$$V \simeq \Sigma p_1^*(V_i) \otimes p_2^*(W_i) \tag{5.14}$$

where the V_i are bundles on X, and the W_i are bundles on $\mathbb{P}^1$. Thus, if $X = \mathbb{A}^N$, such a deformation is globally trivial, since V_i is necessarily trivial. Applying this to $V_{(A,B)}$, we find:

<u>Proposition 5.15</u>. If the pointwise Kronecker invariants are constant, (A,B) is feedback equivalent to a system defined over <u>k</u>.

This has the immediate corollary that such a system is coefficient-assignable, i.e. Theorem 5.2 holds. This latter fact generalizes to arbitrary R, although the Proposition fails to hold in general ([12]). It is not hard to show that, in this case, the pointwise Kronecker invariants are equal to the "global" Kronecker invariants, i.e. the Kronecker invariants computed over the fraction field of R. Indeed, in general, there is a Zariski open set on which the pointwise invariants are constant and equal to the global Kronecker invariants. What is more, the exceptional values must bear some relation to the global invariants. Explicitly, if $V_i = \bigoplus_{i=1}^{m} \mathcal{O}(n_i)$, $V' = \bigoplus_{i=1}^{m'} \mathcal{O}(n_i')$ are bundles on $\mathbb{P}^1$, then $V \geq V'$ in the Harder-Narasimhan ordering just in case

$$\sum_{i=1}^{\ell} n_i \geq \sum_{i=1}^{\ell} n_i' \ , \quad \text{for} \quad \ell = 1,\ldots,\max(m,m'). \tag{5.15}$$

Theorem 5.16 (Shatz, [45]) The Harder-Narasimhan ordering is upper semi-continuous in algebraic families of vector bundles on $\mathbb{P}^1$. This theorem has several applications in problems of computing feedback invariants. Now, any algebraic family of controllable systems induces a map,

$$X \to \tilde{\Sigma}_0,$$

and Theorem 5.16 thus gives a relation between the quotient topology on the finite set $\tilde{\Sigma}_0/\mathcal{F}$ and the Harder-Narasimhan ordering or, equivalently, the Rosenbrock ordering of Theorem 5.11. We call the ordering induced by specialization, the geometric ordering, while we refer to the "ubiquitous ordering" as the natural ordering. The natural ordering is also present in combinatorics and in number theory, where it is studied as an ordering on $\mathcal{L}_n$ - the set of all partitions of the integer n. Now, it is an unpublished piece of folklore, not too hard to prove, that a stronger version of Theorem 5.16 holds for algebraic families constructed via system theory, viz. the geometric ordering is dual to the natural ordering. In, this context, the following theorem ([8]) is rather remarkable:

Proposition 5.17. $\mathcal{L}_n$ is a lattice under the natural ordering.

We close by indicating extensions of this situation which are motivated by control theory but seem likely to contribute some new examples to geometry. First, it is of much more practical significance to study the action of the output feedback group; i.e. $\mathrm{Hom}(k^n,k^m)$ is replaced by $\mathrm{Hom}(k^p,k^m)$ in (iii). In this setting, the proper generalization of Theorem 5.2 is not even conjectured - although a counter-example to the dominant morphism approach has been constructed by J.C. Williems ([48]). Second, pencils may also be defined by control systems governed by partial differential equations. In this setting, the generalization of the Rosenbrock pencil leads to coherent sheaves on $\mathbb{P}^N$ but whether these all split, as in Grothendieck's theorem, is unknown at present. Finally, there are several computations which suggest that, for non-constant coefficient differential equations, the transfer function ought to be, in the scalar input-output case, a meromorphic function on a Riemann surface of higher genus. Each of these questions deserves further study.

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